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LETTER TO THE EDITOR

Scale-dependent multifractal analysis for white noise

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Abstract. We introduce a scale-dependent multifractal analysis which is applicable to any data including non-fractal fluctuations It is found that a white noise can be characterized by a scale-dependent parabolic f–α diagram due to a finite size effect A possible systematic error in the conventional multifractal analysis is discussed

Multifractal analysis [1] was introduced as a kind of generalization of fractal dimensions [2] and is now commonly used for characterizing scale-invariant singularities. Although the method is known to be very powerful, it has been pointed out that a systematic error due to the finiteness of data cannot be neglected in some cases [3]. The systematic error appears when the data do not show a clear scale invariance, so the smallness of data naturally makes the error larger. In order to clarify this finite-size effect, we introduce a scale-dependent multifractal analysis which is applicable to data of any size and to non-fractal fluctuations.

The fractal dimension has already been generalized to be dependent on observation scales since its conception by Mandelbrot by the term 'effective dimension' [1]. His intuitive idea has been formulated and applied to random walks with finite mean-free paths [4]. It is known that the experimentally observed transition of fractal dimension from 1 (linear flight) to 2 (diffusive flight) shows a good agreement with a theoretical curve for the whole range of scales [5], and the scale-dependent fractal dimension has been recognized to be useful for describing some chemical reaction rates [6].

In this letter we analyse scale-dependent multifractality of white noise which should have a trivial fractality in the usual sense. We are going to show that white noise exhibits non-trivial and rich multifractality at finite scales It is found that the normalized variance of the data is an essential quantity to characterize the scale-dependence. In the case that the system size or the number of data is of the same order with the normalized variance the scale-dependence ceases to be apparent and we have a f–α diagram which looks invariant for a few decades of observation scales.

The definition of the scale-dependent multifractal analysis is a natural generalization of the usual multifractal analysis as follows. We define it in terms of probability measure in one-dimensional lattice of size N. Let m_i be the probability of finding a particle at the ith site. We divide the lattice into boxes of size r and define $p_j(r)$ by the sum of $\{m_i\}$ in the jth box. The partition function is introduced as usual by the following equation.

$$Z_q(r) \equiv \sum_{j=1}^{N/r} p_j(r)^q. \tag{1}$$

We introduce a scale-dependent exponent $\tau_q(r)$ by the local gradient of $\log[Z_q(r)]$–$\log[r]$ plot, i.e.

$$\tau_q(r) \equiv \frac{\partial \log Z_q(r)}{\partial \log r} \tag{2}$$

where the derivative is defined by a difference in actual data analysis. In the conventional definition of the multifractal there is no room for allowing such r-dependence explicitly, so the value of $\tau_q(r)$ has been estimated by a linear fit like the least-squares method. Here, $\tau_q(r)$ has an explicit scale-dependence, therefore we can define $\tau_q(r)$ for any data including non-fractal cases.

By using this $\tau_q(r)$, the scale-dependent f–α diagram is defined naturally as

$$\alpha_q(r) \equiv \frac{\partial \tau_q(r)}{\partial q} \tag{3}$$

$$f(\alpha, r) \equiv q\alpha_q(r) - \tau_q(r). \tag{4}$$

It is obvious from these definitions that the f–α diagram is scale-independent if the data's multifractality is perfect like the case of the binomial multifractal [1]. For other cases the f–α diagram become scale-dependent in general, and we can check the validity of the assumption of scale-independence or observe a transition of f–α diagram as a function of r

In the following discussion we analyse random independent fluctuations to know the basic behaviour of this scale-dependent multifractal analysis. We consider the case that $\{m_i\}$ are non-negative identically distributed independent random numbers. Since the system is statistically uniform, (1) is evaluated as

$$Z_q(r) = \frac{N}{r} \langle p_j(r)^q \rangle \tag{5}$$

where $\langle \dots \rangle$ denotes the ensemble average over $\{m_i\}$. For positive integer q, the qth moment of $p_j(r)$ can be expanded as

$$\langle p_j(r)^q \rangle = \langle (m_1 + \ldots + m_r)^q \rangle = \sum_{k=1}^{q} A_k r^k \tag{6}$$

where

$$A_k = \frac{q!}{k!} \sum_{n_1 + \ldots + n_k = q} \frac{\langle m^{n_1} \rangle_c \ldots \langle m^{n_k} \rangle_c}{n_1! \ldots n_k!}. \tag{7}$$

Here the summation is taken over all combination of positive integers $\{n_1, n_2, \ldots, n_k\}$ which satisfies $n_1 + \ldots + n_k = q$, and $\langle m^n \rangle_c$ denotes the nth cumulant. Regarding r as a continuous variable we have $\tau_q(r)$ from (2) as

$$\tau_q(r) = \sum_{k=2}^{q} (k-1) A_k r^{k-1} \Bigg/ \sum_{k=1}^{q} A_k r^k. \tag{8}$$

In the case where high cumulants are negligible, as in the case of a Gaussian distribution, we can approximate the right-hand side of (8) as follows for small q and large r,

$$\tau_q(r) = (q-1)\left(1 - \frac{q}{2} \frac{r_c}{r}\right) \tag{9}$$

where

$$r_c = \frac{\langle m^2 \rangle_c}{\langle m \rangle^2}. \tag{10}$$

Regarding also q as a continuous variable in (9) the f–α relation is obtained from (3) and (4) as

$$f(\alpha, r) = 1 - \frac{1}{2}\frac{r}{r_c}\left(\alpha - 1 - \frac{1}{2}\frac{r_c}{r}\right)^2. \tag{11}$$

The corresponding f–α diagram is a parabolic curve with its centre at $1 + r_c/2r$ and curvature r/r_c. For large r the parabola becomes thinner, and in the limit $r \to \infty$ we have a single point $f = \alpha = 1$, which is consistent with the intuitive estimation that the fractal dimension of a white noise on a line should take a trivial value, 1.

The validity of (11) is confirmed numerically. In figure 1 we plot $\log[Z_q(r)]$ against $\log[r]$ for data of nearly Gaussian positive random numbers, $\{m_i\}$. Here, the system size is $N = 2^{14}$ and $r_c = 0.084$. For small r we can clearly recognize the scale-dependence on the slope of the curves, but for $r/r_c \geq 10^2$ the curves may look linear. This linearity is not genuine, of course; actually we obtain scale-dependent f–α diagrams as shown in figure 2 by taking derivatives as defined by (2). The curves are in good agreement with the theoretical curves (equation (11)) around the top of the parabolas (thin line), which justifies the above theoretical estimates.

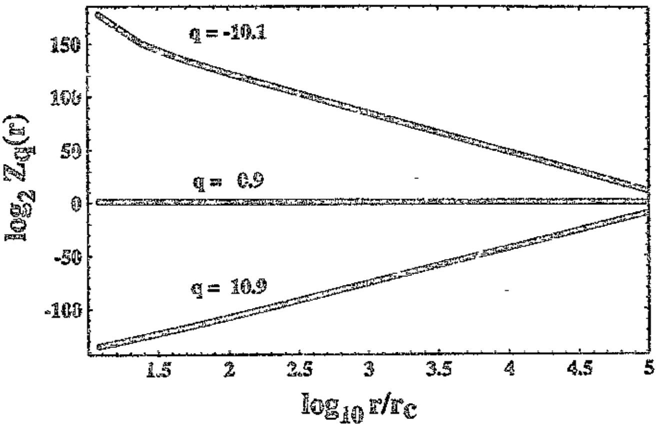

Figure 1. $\text{Log}[Z_q(r)]$–$\log[r]$ plots for three values of q ($q = -10.1, 0.9, 10.9$). The data of $\{m_i\}$ are positive independent random numbers having nearly Gaussian distributions around the mean. The system size is 2^{14} and $r_c = 0.084$.

In order to see the scale-dependence of f–α diagram more clearly we introduce a contour plot of f–α diagram as shown in figure 3. The scale-dependence is much clearer in figure 3 than the commonly used $\log[Z_q(r)]$–$\log[r]$ plot (figure 1).

From the above analysis we found that the observation scale, r, should be normalized by r_c, and the intuitive value for a white noise, $f = \alpha = 1$, can be obtained only for relatively large r, $r/r_c \geq 10^3$. Therefore, the system size, N, must be larger than $r_c \times 10^3$ to confirm the trivial value numerically.

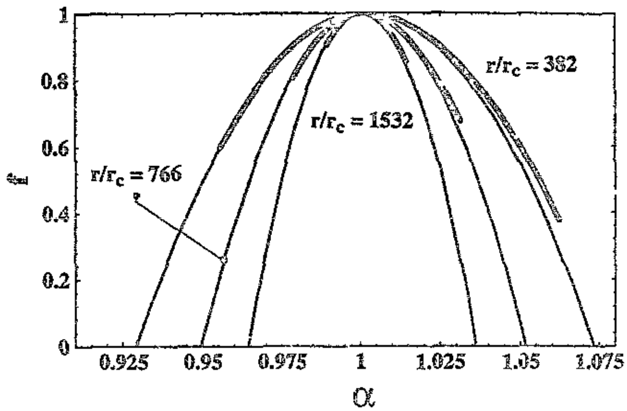

Figure 2. The f–α diagrams for three typical values of r ($r/r_c = 382$, 766 and 1532) The bold lines show the values for the same data as in figure 1 The thin lines are theoretical curves by (11)

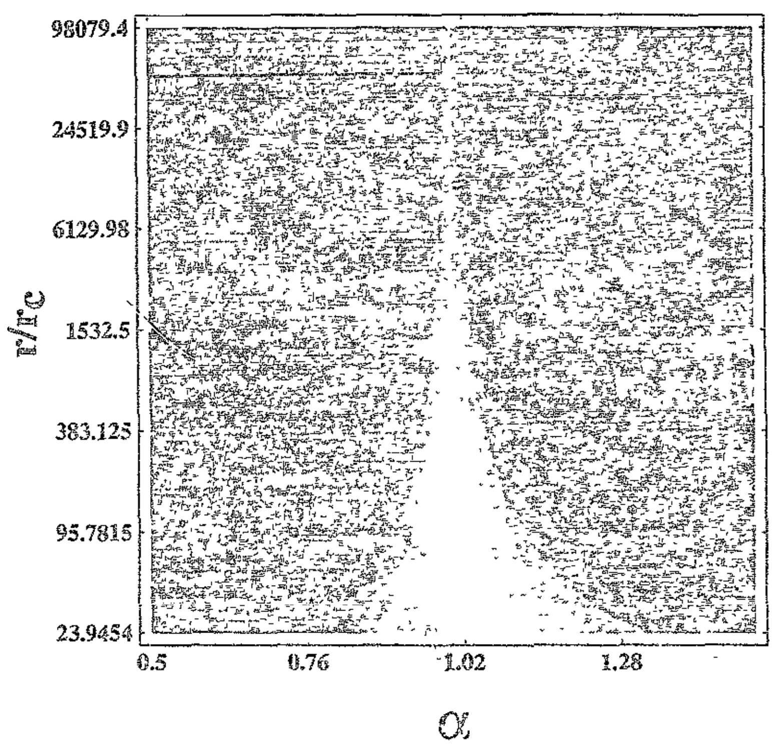

Figure 3. A contour plot of a scale-dependent f–α diagram in a semilog scale The values of f are represented by 10 different shades from black ($0 \leqslant f < 0.1$) to white ($0.9 \leqslant f \leqslant 1$) The data are the same as in figures 1 and 2

For distributions with very large fluctuations the values of r_c can be so large that N and r_c are of the same order. We analyse the case of a white power law distribution as an example In the case of a power law distribution it is known that all higher cumulants, $\langle m^n \rangle_c$, diverge in population, and for finite number of samples $\langle m^n \rangle_c$ become larger as the number of samples are increased. Accordingly the analytical estimation for $\tau_q(r)$ in (8) becomes very complicated, so we treat this case only numerically.

In figure 4 the contour diagram is shown for a power law white noise. The system size is 2^{18} and the distribution function of the probability density is proportional to $m^{-1.8}$. The scale-dependence is much less obvious in this case than in the previous case, figure 3. The highest values of f shift monotonically with increasing r but very slowly Although a rapid shrinking is found for large r comparable with the system size N, the contour looks nearly invariant especially in the region, $\alpha < 1$ and $4 \times 10^{-5} < r/r_c < 4 \times 10^{-2}$.

For a power law distribution without any upper cut-off the normalized variance, r_c, diverges as $N \to \infty$, hence this type of pseudo-scale-invariance in f–α diagram might be observable for any large N. On the other hand if there is an upper cut-off in the distribution tail, then r_c takes a finite value and the contour diagram should behave like the Gaussian case for sufficiently large N

Concluding this letter it may now be obvious that the finiteness of data number can cause a systematic error in f–α diagram if we simply apply the conventional

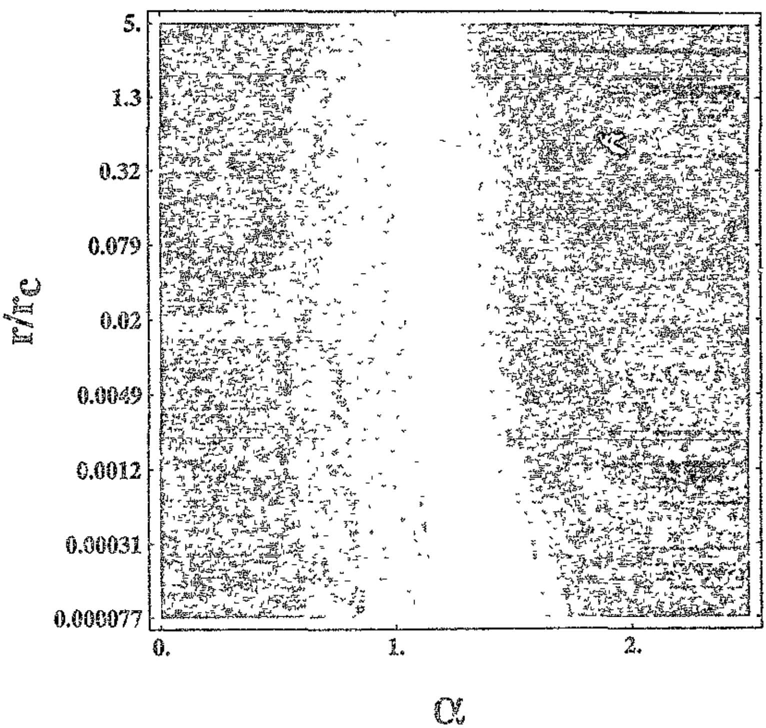

Figure 4. A contour plot of a scale-dependent f–α diagram for a power law white noise The system size is 2^{18} and $r_c = 25\,980$ The probability density for the distribution of $\{m_i\}$ is proportional to $m^{-1.8}$

multifractal analysis for random data. As we have seen in figure 1, the $\log[Z_q(r)]$–$\log[r]$ plot which is usually given as evidence of scale-invariance is not sufficient. In order to treat the finite-size effect more correctly we suggest using a contour f–α diagram such as figure 3 with the observation scale r normalized by r_c. When the system size N and r_c are of the same order, the finite size effect may not be neglected as demonstrated by figure 4, therefore special care should be taken, for example, comparing the contour diagram with that of an artificial uncorrelated random data having the same r_c value.

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